Ejemplo: dado $f(x) = x^2$

a. Halla la STF de f en $[0,2]$

b. Halla la Serie de senos de f en $[0,2]$

c. Halla la serie de cosenos de f en $[0,2]$.

(a) en $L^2[0,2]$, el conjunto $\{1, \cos(n\pi x), \operatorname{sen}(n\pi x)\}_{n=1}^{\infty}$ es ortogonal completo.

$$\langle \cos(n\pi x), \cos(k\pi x)\rangle = \int_0^2 \cos(n\pi x)\cos(k\pi x)\,dx = \frac{1}{2\pi}\left(\frac{\operatorname{sen}(2\pi(n-k))}{n-k} + \frac{\operatorname{sen}(2\pi(n+k))}{n+k}\right) = 0$$

si $n \neq k$

$$\langle \cos(n\pi x), \cos(n\pi x)\rangle = \int_0^2 \cos^2(n\pi x)\,dx = \frac{\operatorname{sen}(4\pi n)}{4\pi n} + 1 = 1$$

$$\langle 1, 1\rangle = \int_0^2 1\,dx = 2$$

$$\langle \cos(n\pi x), \operatorname{sen}(k\pi x)\rangle = \int_0^2 \cos(n\pi x)\operatorname{sen}(k\pi x)\,dx = 0$$

$$\langle \operatorname{sen}(n\pi x), \operatorname{sen}(k\pi x)\rangle = \int_0^2 \operatorname{sen}(n\pi x)\operatorname{sen}(k\pi x)\,dx = 0$$

si $k \neq n$

$$\langle \operatorname{sen}(n\pi x), \operatorname{sen}(n\pi x)\rangle = \int_0^2 \operatorname{sen}^2(n\pi x)\,dx = 1$$

STF en $[0,2]$: $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \operatorname{sen}(n\pi x)$

$$a_n = \frac{1}{1}\int_0^2 f(x)\cos(n\pi x)\,dx = \frac{4}{\pi^2 n^2}, \quad n = 1,2,\ldots \qquad a_0 = \frac{8}{3}$$

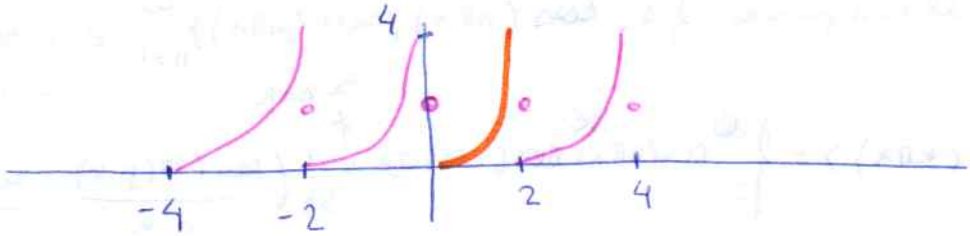
$$b_n = \frac{1}{1}\int_0^2 f(x)\operatorname{sen}(n\pi x)\,dx = -\frac{4}{\pi n}$$

STF: $\boxed{\frac{4}{3} + \sum_{1}^{\infty} \frac{4}{\pi^2 n^2}\cos(n\pi x) - \frac{4}{\pi n}\operatorname{sen}(n\pi x)}$

Convergencia: en $[0,2]$ f es continua por tramos, f' es cont. por tramos $\Rightarrow$ converge a: $f(x)=x^2 \to$ para $x \in (0,2)$

$$\frac{f(0)+f(2)}{2} = 2 \quad \text{para } x=0,2$$

Fuera de $(0,2)$: converge a la extensión periódica (periodo 2) de f y en $x = 2j$ converge a 2.



no converge uniformemente porque el límite no es continuo.

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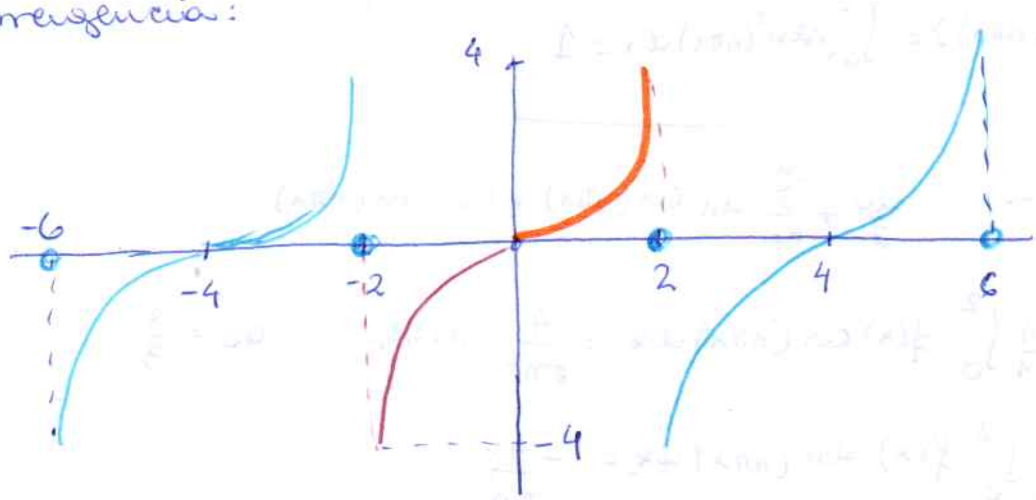
b) Serie seno: $\sum b_n \operatorname{sen}\left(\frac{n\pi x}{2}\right)$ → serie trigon. de Fourier de la extensión impar de f
en $[0,2]$

donde $b_n = \frac{2}{2}\int_0^2 f(x) \operatorname{sen}\left(\frac{n\pi x}{2}\right) dx = \left[\frac{2((-1)^n-1)}{\pi^3 n^3} - \frac{1}{\pi n}(-1)^n\right] 8$

Convergencia:



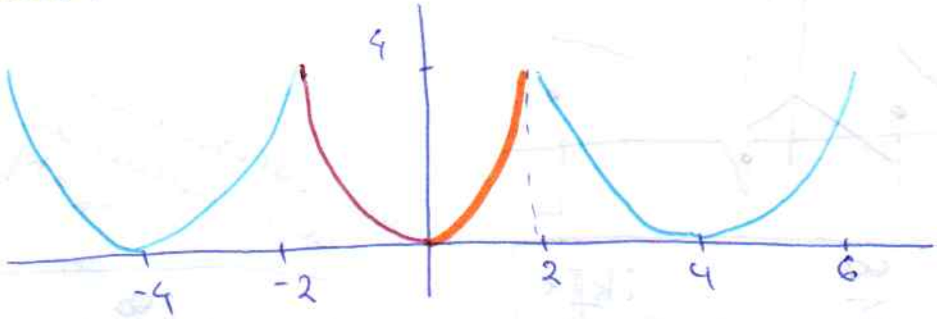
converge puntualmente a la extensión impar periódica (periodo 4) de f, y a 0 en los puntos $x = 2j$

c) Serie de cosenos: $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$
en $[0,2]$

donde $a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \frac{16}{\pi^2 n^2}(-1)^n$ si $n \neq 0$

$a_0 = \frac{8}{3}$

Convergencia:



converge puntualmente a la extensión par periódica, periodo 4.

convergencia uniforme.

Notar:

$$S(x) = \frac{4}{3} + \sum \underbrace{\frac{16(-1)^n}{\pi^2 n^2}\cos\left(\frac{n\pi x}{2}\right)}_{g_n(x)}$$

$|g_n(x)| \leq \frac{16}{\pi^2 n^2}$ para todo $x \in \mathbb{R}$.

y $\sum_1^{\infty} \frac{16}{\pi^2 n^2}$ converge $\Rightarrow \sum_1^{\infty} g_n(x)$ converge uniformemente en $\mathbb{R}$.

Criterio de Weierstrass

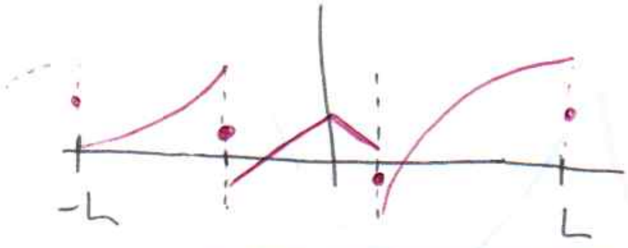
Con $x=0$, $S(0)$ converge a $f(0)=0$.

$$S(0) = \frac{4}{3} + \sum \frac{16}{\pi^2 n^2}(-1)^n = 0 \Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{4}{3}\cdot\frac{1}{16}\pi^2 = -\frac{\pi^2}{12}}$$

Hacia transformada de Fourier.

Dada $f:[-L,L]\to\mathbb{C}$, f y f' continua por tramos; y supongamos que $f(x)=\dfrac{f(x^-)+f(x^+)}{2}$ para todo $x\in(-L,L)$

(así, la SF. converge a f en todo punto) de $(-L,L)$



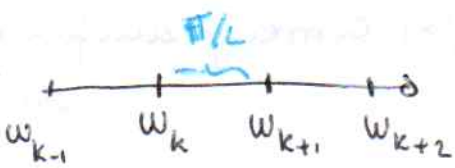
suma de oscilaciones de frecuencias discretas $\omega_k=\frac{k\pi}{L}$

$$\boxed{f(x)=\sum_{k=-\infty}^{\infty} c_k e^{i\frac{k\pi}{L}x}} \quad \text{con} \quad \boxed{c_k=\frac{1}{2L}\int_{-L}^{L} f(t)e^{-i\frac{k\pi}{L}t}\,dt}$$

$$f(x)=\sum_{k=-\infty}^{\infty}\underbrace{\left(\frac{1}{2L}\int_{-L}^{L} f(t)e^{-i\frac{k\pi}{L}t}\,dt\right)}_{c_k}\cdot e^{i\frac{k\pi}{L}x}$$

Sea $\omega_k=\frac{k\pi}{L} \rightarrow \omega_{k+1}-\omega_k=\Delta\omega_k=\frac{\pi}{L} \Rightarrow \frac{1}{2L}=\frac{\Delta\omega_k}{2\pi}$

$$f(x)=\sum_{k=-\infty}^{\infty}\left(\int_{-L}^{L} f(t)e^{-i\omega_k t}\,dt\right)\cdot e^{i\omega_k x}\cdot\frac{\Delta\omega_k}{2\pi}$$



con $L\to\infty$, $\Delta\omega_k\to 0$, ω_k se hace densa en $\mathbb{R}$.

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt}_{\hat f(\omega)=\mathcal{F}(f)(\omega)}\cdot e^{i\omega x}\cdot d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat f(\omega)\cdot e^{i\omega x}\,d\omega$$

"suma" de oscilaciones con frecuencias continuas ω

Definición

Dada $f:\mathbb{R}\to\mathbb{C}$, si $\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$ converge,

se llama TRANSFORMADA DE FOURIER de f.

$$\mathcal{F}(f)(\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt \qquad (*)$$

Condición suficiente para la existencia de T.F.:

si f es absolutamente integrable en $\mathbb{R}$ (es decir, si $\int_{-\infty}^{\infty}|f(t)|\,dt<\infty$) entonces existe la T.F. de f.

Se deduce de: $|f(t)e^{-i\omega t}| = |f(t)|$

si $\int_{-\infty}^{\infty}|f(t)|\,dt<\infty \Rightarrow \int_{-\infty}^{\infty}|f(t)e^{-i\omega t}|\,dt<\infty$ y $(*)$ converge absolutamente

entonces, converge.

Def: $\boxed{L^1(\mathbb{R})}$: funciones $f:\mathbb{R}\to\mathbb{C}$ absolutamente integrables en $\mathbb{R}$.

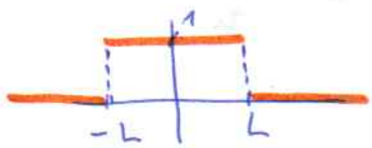
Ejemplos.

(A) $\boxed{f(x)=e^{-a|x|}}$, $a>0$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-a|x|}\cdot e^{-i\omega x}\,dx = \int_{-\infty}^{0} e^{(a-i\omega)x}\,dx + \int_{0}^{\infty} e^{(-a-i\omega)x}\,dx$$

$$= \frac{e^{(a-i\omega)x}}{a-i\omega}\Big|_{-\infty}^{0} + \frac{e^{(-a-i\omega)x}}{-a-i\omega}\Big|_{0}^{\infty} = \frac{1}{a-i\omega}+\frac{1}{a+i\omega} = \boxed{\frac{2a}{a^2+\omega^2}}$$

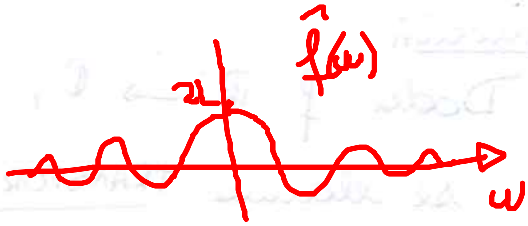
(B) $\boxed{f(x)=\begin{cases}1 & |x|<L\\ 0 & |x|>L\end{cases}}$



$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx = \int_{-L}^{L} e^{-i\omega x}\,dx \overset{\omega\neq 0}{=} \frac{e^{-i\omega x}}{-i\omega}\Big|_{-L}^{L} = \frac{e^{-i\omega L}-e^{i\omega L}}{-i\omega}$$

$$= 2\,\frac{\operatorname{sen}(\omega L)}{\omega} \quad \text{si } \omega\neq 0$$

$$\hat{f}(0) = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-L}^{L} dx = 2L$$



$$\boxed{\hat{f}(\omega) = \begin{cases} \dfrac{2\,\mathrm{sen}(\omega L)}{\omega} & \text{si } \omega \neq 0 \\ 2L & \text{si } \omega = 0 \end{cases}}$$

→ resulta continua en $\mathbb{R}$. y real

ⓒ
$$\boxed{f(x) = \begin{cases} \mathrm{sen}(x) & 0 < x < \pi \\ 0 & \text{en otro caso.} \end{cases}}$$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \int_0^{\pi} \mathrm{sen}(x) . e^{-i\omega x} dx = \int_0^{\pi} \left(\frac{e^{ix} - e^{-ix}}{2i}\right) e^{-i\omega x} dx$$

$$= \int_0^{\pi} \frac{e^{ix(1-\omega)}}{2i} - \frac{e^{ix(-1-\omega)}}{2i}\, dx \underset{\uparrow}{=} \frac{1}{2i}\left[\frac{e^{ix(1-\omega)}}{i(1-\omega)}\Big|_0^{\pi} - \frac{e^{ix(-1-\omega)}}{i(-1-\omega)}\Big|_0^{\pi} \right] =$$

$\omega \neq \pm 1$

$$= -\frac{1}{2}\left[\frac{e^{i\pi(1-\omega)} - 1}{(1-\omega)} + \frac{e^{i\pi(-1-\omega)} - 1}{1+\omega} \right] = -\frac{1}{2}\left[\frac{(-1).e^{-i\pi\omega}}{1-\omega} + \frac{(-1)\,e^{-i\pi\omega}}{1+\omega} - \frac{1}{1-\omega} - \frac{1}{1+\omega} \right] =$$

$$= \frac{e^{-i\pi\omega}}{2} \frac{2}{(1-\omega^2)} - \frac{1}{(1-\omega^2)} = \frac{e^{-i\pi\omega} - 1}{1-\omega^2} \quad \text{si } \omega \neq \pm 1$$

$$\hat{f}(1) = \int_0^{\pi} \mathrm{sen}(x) . e^{-ix} dx = \int_0^{\pi} \frac{e^{ix} - e^{-ix}}{2i} e^{-ix} dx = \int_0^{\pi} \frac{1 - e^{-2ix}}{2i} dx =$$

$$= \frac{\pi}{2i} - \frac{e^{-2ix}}{4}\Big|_0^{\pi} = -\frac{i\pi}{2}$$

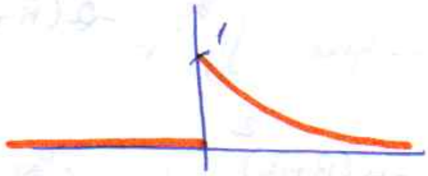
$$\hat{f}(-1) = \int_0^{\pi} \mathrm{sen}(x) e^{ix} dx = \int_0^{\pi} \frac{e^{ix} - e^{-ix}}{2i} e^{ix} dx = \int_0^{\pi} \frac{e^{2ix} - 1}{2i} dx = \frac{e^{2ix}}{-4}\Big|_0^{\pi} - \frac{\pi}{2i}$$

$$= \frac{i\pi}{2}.$$

$$\boxed{\hat{f}(\omega) = \begin{cases} \dfrac{e^{-i\pi\omega} - 1}{1-\omega^2} & \text{si } \omega \neq 1, -1 \\ -\dfrac{i\pi}{2} & \omega = 1 \\ \dfrac{i\pi}{2} & \omega = -1 \end{cases}}$$

→ resulta continua. (no real)

D) $$\boxed{f(x)=\begin{cases} e^{-x} & x>0 \\ 0 & x<0 \end{cases}}$$



$$\hat{f}(\omega)=\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx=\int_0^{\infty} e^{-x}e^{-i\omega x}dx=\left.\frac{e^{-x(1+i\omega)}}{-(1+i\omega)}\right|_0^{\infty}=$$

$$\boxed{\hat{f}(\omega)=\frac{1}{1+i\omega}}$$ → es compleja

E) $$\boxed{f(x)=e^{-ax^2}} \qquad a>0$$

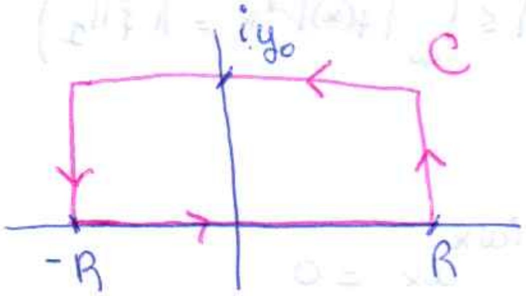
$$\hat{f}(\omega)=\int_{-\infty}^{\infty} e^{-ax^2}e^{-i\omega x}dx=\int_{-\infty}^{\infty} e^{-a\left(x^2+\frac{i\omega}{a}x\right)}dx=$$

$$=\int_{-\infty}^{\infty} e^{-a\left(x+\frac{i\omega}{2a}\right)^2-\frac{\omega^2}{4a}}dx=e^{-\frac{\omega^2}{4a}}\int_{-\infty}^{\infty} e^{-a\left(x+\frac{i\omega}{2a}\right)^2}dx \overset{u=x+\frac{i\omega}{2a}}{=}$$

$$=e^{-\frac{\omega^2}{4a}}\underbrace{\int_{-\infty+\frac{i\omega}{2a}}^{\infty+\frac{i\omega}{2a}} e^{-au^2}du}_{I}=e^{-\frac{\omega^2}{4a}}\cdot I=e^{-\frac{\omega^2}{4a}}\cdot\sqrt{\frac{\pi}{a}}$$

clase 20, pág 10-11

Veremos que $I=\int_{-\infty}^{\infty} e^{-ax^2}dx=\sqrt{\frac{\pi}{a}}$



$$\int_C e^{-az^2}dz=0$$

$$\underbrace{\int_{-R}^{R} e^{-ax^2}dx}_{\substack{z=x\\dz=dx}}+\underbrace{\int_0^{y_0} e^{-a(R+iy)^2}i\,dy}_{\substack{z=R+iy\\dz=i\,dy}}-\underbrace{\int_{-R}^{R} e^{-a(x+iy_0)^2}dx}_{\substack{z=x+iy_0\\dz=dx}}-\underbrace{\int_0^{y_0} e^{-a(-R+iy)^2}i\,dy}_{\substack{z=-R+iy\\dz=i\,dy}}=0$$

$\xrightarrow{R\to\infty} 0$ $\qquad\qquad\qquad\qquad$ $\xrightarrow{R\to\infty} 0$

Veamos que $\int_0^{y_0} e^{-a(R+iy)^2}\,dy \xrightarrow[]{R\to\infty} 0$

$$\left|\int_0^{y_0} e^{-a(R+iy)^2}\,dy\right| \leq \int_0^{y_0} \left|e^{-a(R+iy)^2}\right| dy = \int_0^{y_0} e^{-a(R^2-y^2)}\,dy =$$

$$= e^{-aR^2}\int_0^{y_0} e^{ay^2}\,dy \xrightarrow[R\to\infty]{} 0$$

Similarmente: $\int_0^{y_0} e^{-a(-R+iy)^2}\,dy \xrightarrow[R\to\infty]{} 0$.

Luego:

$$\overbrace{\int_{-\infty}^{\infty} e^{-ax^2}\,dx}^{\sqrt{\frac{\pi}{a}}} - \int_{-\infty}^{\infty} e^{-a(x+iy_0)^2}\,dx = 0 \quad \text{para cq } y_0.$$

TP 8
2h

La transformada buscada:

$$\boxed{\hat{f}(\omega) = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}}}$$

Primeros resultados sobre T.F.

Teorema:

Si $f \in L^1(\mathbb{R})$, entonces:

a) $\hat{f}(\omega)$ existe y es acotada $\left(|\hat{f}(\omega)| \leq \int_{-\infty}^{\infty} |f(x)|\,dx = \overbrace{\|f\|_1}^{\text{norma 1}} \right)$

b) $\hat{f}$ es continua en $\mathbb{R}$ y ~~para~~

c) $\lim_{\omega\to\pm\infty} \hat{f}(\omega) = \underbrace{\lim_{\omega\to\pm\infty} \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx}_{\text{recordar propiedad simil p/ SF.}} = 0$

Si $f \notin L^1(\mathbb{R})$, $\hat{f}$ podría no ser continua:

$f(x) = \frac{\operatorname{sen}(x)}{x}$ → no es Absolutamente integrable.

Pero $\hat{f}(\omega) = \int_{-\infty}^{\infty} \frac{\operatorname{sen}(x)}{x} e^{-i\omega x} dx = \underbrace{\int_{-\infty}^{\infty} \frac{\operatorname{sen}(x)}{x} \cdot \cos(\omega x) - i \frac{\operatorname{sen} x \operatorname{sen}(\omega x)}{x} dx}_{\text{converge por criterio Dirichlet-Abel}}$

$\hat{f}(\omega) = \int_{-\infty}^{\infty} \frac{\operatorname{sen}(x)}{x} \cos(\omega x) dx$ → es real.

Calculemos: $\hat{f}(\omega) = \int_{-\infty}^{\infty} \frac{e^{ix} - e^{-ix}}{2ix} e^{-i\omega x} dx = \int_{-\infty}^{\infty} \frac{e^{ix(1-\omega)} - e^{ix(-1-\omega)}}{2ix} dx$

$= \frac{1}{2i} VP \int_{-\infty}^{\infty} \frac{e^{ix(1-\omega)}}{x} dx - \frac{1}{2i} VP \int_{-\infty}^{\infty} \frac{e^{ix(-1-\omega)}}{x} dx$

Recordemos: $VP \int_{-\infty}^{\infty} \frac{e^{iax}}{x} dx = \underbrace{VP \int_{-\infty}^{\infty} \frac{\cos(ax)}{x} dx}_{=0} + i\, VP \int_{-\infty}^{\infty} \frac{\operatorname{sen}(ax)}{x} dx \overset{a>0}{=} i\pi$

clase 20, pág 6-7

$VP \int_{-\infty}^{\infty} \frac{e^{iax}}{x} dx \overset{a<0}{=} -i\pi$

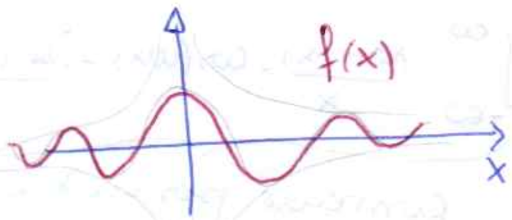
Entonces:

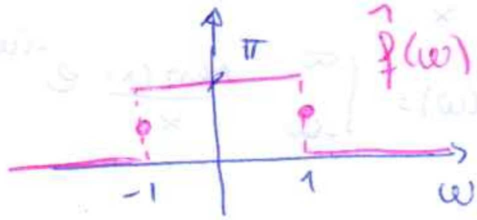
$$\hat{f}(\omega) = \begin{cases} - \text{si } (1-\omega) < 0 \wedge (-1-\omega) < 0 \iff 1 < \omega : \frac{1}{2i}(-i\pi + i\pi) = 0 \\ - \text{si } (1-\omega) < 0 \wedge (-1-\omega) > 0 \to \text{imposible} \\ - \text{si } (1-\omega) > 0 \wedge (-1-\omega) > 0 \iff \omega < -1 : \frac{1}{2i}(i\pi - i\pi) = 0 \\ - \text{si } (1-\omega) > 0 \wedge (-1-\omega) < 0 \iff -1 < \omega < 1 : \frac{1}{2i}(i\pi - (-i\pi)) = \pi \\ - \text{si } \omega = 1 \text{ o } \omega = -1 : \quad \pi/2 \end{cases}$$

Cálculo: $\omega = 1$:

$\hat{f}(1) \overset{VP}{=} \int_{-\infty}^{\infty} \frac{1 - e^{-ix.2}}{2ix} dx = \frac{1}{2i} VP \int_{-\infty}^{\infty} \frac{1}{x} dx - \frac{1}{2i} VP \int_{-\infty}^{\infty} \frac{e^{-i2x}}{x} dx = 0 - \frac{1}{2i}(-i\pi)$

$$\hat{f}(\omega) = \begin{cases} 0 & |\omega| > 1 \\ \pi & |\omega| < 1 \\ \pi/2 & |\omega| = 1 \end{cases}$$





Más propiedades.

(A) Si $f(x)$ es real, entonces:

- $\overline{\hat{f}(\omega)} = \hat{f}(-\omega)$

Dem: $\overline{\hat{f}(\omega)} = \overline{\int f(x) e^{-i\omega x} dx} = \int f(x) \overline{e^{-i\omega x}} dx = \int f(x) e^{i\omega x} dx$

$= \int f(x) . e^{-i(-\omega)x} dx = \hat{f}(-\omega)$

Prob 6 TP 8

- si f es par $\Rightarrow \hat{f}(\omega)$ es real y par

Dem: $\hat{f}(\omega) = \int f(x) e^{-i\omega x} dx = \int f(x) \cos(\omega x) dx - i \underbrace{\int f(x) \operatorname{sen}(\omega x) dx}_{=0}$

$= \int f(x) \cos(\omega x) dx = \int f(x) \cos(-\omega x) dx =$

$= \hat{f}(-\omega)$

- si f es impar $\Rightarrow i\hat{f}(\omega)$ es real e impar

Dem: se deja al lector.

(B) La transformada es lineal: si f y g tienen T.F,

- $\widehat{f+g} = \hat{f} + \hat{g}$
- $\widehat{\lambda f} = \lambda \hat{f} \quad \lambda \in \mathbb{C}$

(C) Traslación. Sea $h_a(x) = f(x-a)$, y $g_a(x) = e^{iax} f(x)$

Entonces $\hat{h}_a(\omega) = e^{-ia\omega} \hat{f}(\omega)$

y $\hat{g}_a(\omega) = \hat{f}(\omega - a)$

Dem: $\widehat{h_a}(\omega) = \int_{-\infty}^{\infty} f(x-a)e^{-i\omega x}dx \underset{t=x-a}{=} \int_{-\infty}^{\infty} f(t)e^{-i\omega(t+a)}dt =$

$$= e^{-i\omega a}\int f(t)e^{-i\omega t}dt = e^{-i\omega a}\hat{f}(\omega)$$

$$\hat{g}_a(\omega) = \int e^{iax}f(x)e^{-i\omega x}dx = \int f(x).\,e^{-i(\omega-a)x}dx = \hat{f}(\omega-a)$$

(D) Sea $g(x) = f(x)\cos(ax)$ y $h(x) = f(x)\,\mathrm{sen}(ax)$

Entonces: $\hat{g}(\omega) = \dfrac{\hat{f}(\omega-a)+\hat{f}(\omega+a)}{2}$; $\hat{h}(\omega) = \dfrac{\hat{f}(\omega-a)-\hat{f}(\omega+a)}{2i}$

Dem: $g(x) = \dfrac{f(x)e^{iax}+f(x)e^{-iax}}{2}$

$\Rightarrow$ por linealidad y traslación:

$$\hat{g}(\omega) = \frac{\hat{f}(\omega-a)+\hat{f}(\omega+a)}{2}$$

La otra transformada se demuestra similarmente.

(E) Sea f derivable y f y $f' \in L^1(\mathbb{R})$, y $\lim_{x\to\pm\infty} f(x) = 0$

$$\widehat{f'}(\omega) = \mathcal{F}(f')(\omega) = i\omega\hat{f}(\omega)$$

Dem: $\hat{f'}(\omega) = \int f'(x)e^{-i\omega x}dx = f(x)e^{-i\omega x}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x).(-i\omega)e^{-i\omega x}dx$

$$= i\omega\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = i\omega\hat{f}(\omega)$$

Nota: se cumple aún cuando f continua, f' continua por tramos, $f, f' \in L^1(\mathbb{R})$

Generalización: $\widehat{f^{(n)}}(\omega) = (i\omega)^n\hat{f}(\omega)$

(F) Si $f(x)$ y $x f(x) \in L^1(\mathbb{R})$, entonces $\hat{f}$ es derivable y

$$\hat{f}'(\omega) = \mathcal{F}(f)'(\omega) = \frac{d\hat{f}(\omega)}{d\omega} = -i\,\widehat{x f(x)}(\omega) = -i\,\mathcal{F}(x f(x))(\omega)$$

Dem:

$$\frac{d}{d\omega}\hat{f}(\omega) = \frac{d}{d\omega}\int f(x)e^{-i\omega x}dx \overset{\text{las hipótesis permiten esto}}{=} \int f(x)\frac{d}{d\omega}e^{-i\omega x}dx = \int f(x)\cdot(-ix)e^{-i\omega x}dx$$

$$= -i\int_{-\infty}^{\infty} x f(x)e^{-i\omega x}dx = -i\,\mathcal{F}(x f(x))(\omega)$$

Generalización: $\widehat{x^n f(x)}(\omega) = i^n \hat{f}^{(n)}(\omega)$

(G) $g_a(x) = f(ax)$. Entonces $\hat{g}_a(\omega) = \frac{1}{|a|}\hat{f}\left(\frac{\omega}{a}\right)$

Dem:

$$\hat{g}_a(\omega) = \int f(ax)e^{-i\omega x}dx \underset{\substack{t=ax \\ \text{Si } a>0}}{=} \int_{-\infty}^{\infty} f(t)e^{-i\omega\frac{t}{a}}\cdot\frac{1}{a}dt = \frac{1}{a}\hat{f}\left(\frac{\omega}{a}\right)$$

si $a<0$:

$$\hat{g}_a(\omega) = -\int_{-\infty}^{\infty} f(t)e^{-i\omega\frac{t}{a}}\frac{1}{a}dt = -\frac{1}{a}\hat{f}\left(\frac{\omega}{a}\right) = \frac{1}{|a|}\hat{f}\left(\frac{\omega}{a}\right)$$

Resumen de propiedades operativas

$$f+g \overset{\mathcal{F}}{\rightsquigarrow} \hat{f}+\hat{g}$$

$$\lambda f \rightsquigarrow \lambda\hat{f}$$

$$f(x-a) \rightsquigarrow e^{-ia\omega}\hat{f}(\omega)$$

$$e^{iax}f(x) \rightsquigarrow \hat{f}(\omega-a)$$

$$f'(x) \rightsquigarrow i\omega\hat{f}(\omega)$$

$$f^{(n)}(x) \rightsquigarrow (i\omega)^n\hat{f}(\omega)$$

$$x f(x) \rightsquigarrow i\hat{f}'(\omega)$$

$$x^n f(x) \rightsquigarrow i^n \hat{f}'(\omega)$$

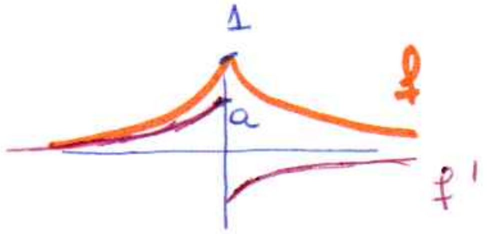
Ejemplos

1) $f(x) = e^{-ax^2}$

$$\underbrace{f'(x) = -2ax\, e^{-ax^2}}_{\text{impar}} \xrightarrow[\text{prop. E}]{\tilde{\mathcal{F}}} i\omega \hat{f}(\omega) = \underbrace{i\omega \cdot \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}}}_{\substack{\text{imag. puro} \\ \text{impar}}}$$

2) $f(x) = e^{-a|x|}$ $\quad a>0$

$$f'(x) = \begin{cases} a e^{ax} & \text{si } x<0 \\ -a e^{-ax} & \text{si } x>0 \end{cases}$$



f continuo, f' cont por tramos, ambos $L^1(\mathbb{R})$, $\lim_{x\to\pm\infty} f = 0$

$$\underbrace{f'(x)}_{\text{impar}} \xrightarrow{\tilde{\mathcal{F}}} i\omega \hat{f}(\omega) = \underbrace{\frac{i\omega \cdot 2a}{a^2+\omega^2}}_{\substack{\text{imag puro,} \\ \text{impar}}} \longrightarrow \notin L^1(\mathbb{R})$$

3) $f(x) = \begin{cases} 1 & a<x<b \\ 0 & \text{en otro caso.} \end{cases}$ $\qquad g(x) = \begin{cases} 1 & -1<x<1 \\ 0 & \text{otro caso.} \end{cases}$

(ejemplo B, pág 5)

$$f(x) = g\left(\left[x - \left(\frac{a+b}{2}\right)\right]\frac{2}{b-a}\right)$$

$$x = a \rightarrow \left[x - \left(\frac{a+b}{2}\right)\right] \cdot \frac{2}{b-a} = -1$$

$$x = b \rightarrow \left[x - \left(\frac{a+b}{2}\right)\right] \frac{2}{b-a} = 1$$

$$f(x) = G\left(x - \left(\frac{a+b}{2}\right)\right)$$

$$G(t) = g\left(t \cdot \frac{2}{(b-a)}\right).$$

$$\hat{f}(\omega) = e^{-i\left(\frac{a+b}{2}\right)\omega} \cdot \hat{G}(\omega) = e^{-i\left(\frac{a+b}{2}\right)\omega} \cdot \frac{1}{2/(b-a)} \cdot \hat{g}\left(\frac{\omega(b-a)}{2}\right)$$

$$= e^{-i\left(\frac{a+b}{2}\right)\omega} \cdot \frac{\cancel{b-a}}{\cancel{2}} \cdot 2 \cdot \frac{\operatorname{sen}(\omega(b-a)/2)}{\omega\,\cancel{\left(\frac{b-a}{2}\right)}} = e^{-i\left(\frac{a+b}{2}\right)\omega} \cdot 2\, \frac{\operatorname{sen}(\omega(b-a)/2)}{\omega}$$

o mejor por definición!

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \int_{a}^{b} e^{-i\omega x} dx = \frac{e^{-i\omega x}}{-i\omega}\Big|_a^b = \frac{(e^{-i\omega b} - e^{-i\omega a})}{\omega} i$$